TABLE II

| | $TM_{101}$ | | | | $TM_{102}$ | | | |
|---|---|---|---|---|---|---|---|---|
| | $Q_{ex}$ | $Q_{pert}$ | $(ka)_{ex}$ | $2a/\lambda_0$ | $Q_{ex}$ | $Q_{pert}$ | $(ka)_{ex}$ | $2a/\lambda_0$ |
| $\epsilon_r$ = 14 | 6 | 4.1 | 4.21 | 0.358 | 11.3 | 7.9 | 7.69 | 0.655 |
| 50 | 84 | 97 | 4.38 | 0.210 | 23.7 | 19 | 7.59 | 0.364 |
| 86 | 330 | 375 | 4.43 | 0.152 | 62.5 | 74 | 7.61 | 0.261 |

$$\bar{p}_e = j(4\pi/Nc)ka^2 \sin ka\bar{u}_z$$

$$Q = N^5/2k_m^3a^3. \tag{70}$$

The values $ka = 4.49$ and $ka = 7.73$ yield, respectively, the $TM_{101}$ and $TM_{102}$ modes of Gastine, for which the results shown in Table II hold.

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# The Excitation of Dielectric Resonators of Very High Permittivity

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***Abstract*—The response of a dielectric resonator excited by either interior volume sources or incident exterior waves is investigated. Special attention is devoted to phenomena at resonance, and in particular to the induced electric and magnetic dipoles. Simple formulas are obtained for the scattering cross section. The material of the resonator is assumed lossless and of very high permittivity.**

## I. INTRODUCTION

IN A PRECEDING article [1] we have investigated the nature and properties of the modes of a dielectric resonator of very high permittivity. In the present paper we make use of the modal properties, and in particular of the orthogonality relationships, to investigate the excitation of a resonator by interior volume sources or, more realistically, by exterior incident fields. Our general method of attack is to assume that the index of refraction $N$ of the (lossless) dielectric is large, and to expand the fields as

$$\bar{E} = \bar{E}_0 + \frac{\bar{E}_1}{N} + \frac{\bar{E}_2}{N^2} + \cdots$$

$$\bar{H} = \bar{H}_0 + \frac{\bar{H}_1}{N} + \frac{\bar{H}_2}{N^2} + \cdots. \tag{1}$$

These expansions are inserted in Maxwell's equations, and terms of equal orders on both sides of these equations are equated. The mechanics of the procedure will be described in subsequent paragraphs. Our main purpose is to determine the dominant terms in (1), and in particular the behavior of these terms in the vicinity of a resonance $k = k_m$. In the limit $N \to \infty$, the magnetic field $\bar{H}_0$ near resonance must be proportional to the relevant eigenvector $\bar{H}_m$, solution of [1],

$$-\text{curl curl } \bar{H}_m + k_m^2\bar{H}_m = 0 \quad \text{in } V$$

$$\text{curl } \bar{H}_m = 0 \quad \text{in } V'. \tag{2}$$

These eigenvectors satisfy the important orthogonality properties [1]

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$$\iiint_{V+V'} \bar{H}_m \cdot \bar{H}_p \, dV = 0$$

$$\iiint_V \text{curl } \bar{H}_m \cdot \text{curl } \bar{H}_p \, dV = 0. \qquad (3)$$

## II. EXCITATION OF A DIELECTRIC RESONATOR BY INTERIOR SOURCES—GENERAL FORMULAS

The configuration of interest is depicted in Fig. 1. It is shown in [1] that the expansion for $\bar{E}$ contains only odd terms in $1/N$, hence that the dominant term of the expansion is $\bar{E}_1/N$. We shall expand $\bar{E}_1$ in the (orthogonal) set curl $\bar{H}_m$. This set is solenoidal, and should be completed by irrotational elements.[1] The latter, however, do not give rise to resonance phenomena [2]. We shall therefore keep only solenoidal terms in the expansion for $\bar{E}_1$, and write the following expression (which represents the solenoidal part of $\bar{E}_1$)

$$(\bar{E}_1)_{\text{sol}} = \sum \alpha_m \text{ curl } \bar{H}_m = \sum \alpha_m \bar{A}_m \qquad \text{in } V \qquad (4)$$

where

$$\alpha_m = \frac{\iiint_V \bar{E}_1 \cdot \text{curl } \bar{H}_m \, dV}{\iiint_V |\text{ curl } \bar{H}_m|^2 \, dV}. \qquad (5)$$

To evaluate the numerator, consider the basic differential equation for $\bar{E}_1$, which can easily be derived from Maxwell's equations

$$-\text{curl curl } \bar{E}_1 + k^2 \bar{E}_1 = jkR_0 \bar{J}. \qquad (6)$$

Here $k$ is the wavenumber in the dielectric, and $R_0 = 120\pi\Omega$ is the characteristic resistance of vacuum. Dot-multiplying (6) with $\bar{A}_m = \text{curl } \bar{H}_m$ gives, after integration over $V$,

$$-\iiint_V \bar{A}_m \cdot \text{curl curl } \bar{E}_1 \, dV + k^2 \iiint_V \bar{A}_m \cdot \bar{E}_1 \, dV = jkR_0 \iiint_V \bar{J} \cdot \bar{A}_m \, dV. \qquad (7)$$

[1] The irrotational elements, which should be included in regions where real charges are present, are of the form grad $\phi_n$, where

$$\nabla^2 \phi_n + k_n^2 \phi_n = 0 \qquad \text{in } V$$
$$\nabla^2 \phi_n = 0 \qquad \text{in } V'$$

$\phi_n$ continuous on $S$

$\phi_n$ regular at infinity (i.e., of order $1/R$).

It is easy to show that grad $\phi_n$ is orthogonal to $\bar{H}_m$ and curl $\bar{H}_m$, in the sense that

$$\iiint_{V+V'} \bar{H}_m \cdot \text{grad } \phi_n \, dV = 0$$

$$\iiint_V \text{curl } \bar{H}_m \cdot \text{grad } \phi_n \, dV = 0.$$

It is also easy, making use of [1, eq. 20] to calculate the expansion coefficient of $\bar{E}_1$ in terms of grad $\phi_n$, and to show that it does not evidence any resonance properties.

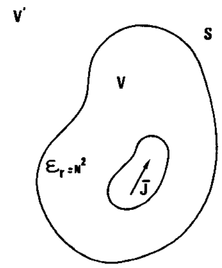


Fig. 1. Dielectric resonator with interior sources.

The first integral can be transformed as follows

$$I = -\iiint_V \bar{A}_m \cdot \text{curl curl } \bar{E}_1 \, dV$$
$$= -\iiint_V \bar{E}_1 \cdot \text{curl curl } \bar{A}_m \, dV + \iint_S [(\bar{u}_n \times \bar{A}_m) \cdot \text{curl } \bar{E}_1 - (\bar{u}_n \times \bar{E}_1) \cdot \text{curl } \bar{A}_m] \, dS. \qquad (8)$$

From (2), and because (see [1, eq. 3])

$$\text{curl } \bar{E}_1 = -jkR_0 \bar{H}_0 \qquad (9)$$

we find

$$I = -k_m^2 \iiint_V \bar{E}_1 \cdot \bar{A}_m \, dV - jkR_0 \iint_S (\bar{u}_n \times \text{curl } \bar{H}_m) \cdot \bar{H}_0 \, dS + j \frac{k_m^2 R_0}{k} \iint_S \bar{H}_m \cdot (\bar{u}_n \times \text{curl } \bar{H}_0) \, dS. \qquad (10)$$

The surface integrals in the second member can be transformed by the methods utilized in [1, eqs. 18–24]. Detailed calculations, not to be given here in detail, show that these integrals cancel each other. We can now combine (7) and (10) to obtain

$$\iiint_V \bar{A}_m \cdot \bar{E}_1 \, dV = \frac{jkR_0 \iiint_V \bar{J} \cdot \bar{A}_m \, dV}{k^2 - k_m^2} \qquad (11)$$

from which we deduce the expansion

$$\bar{E}_1 = N\bar{E} = jkR_0 \sum \frac{\iiint_V \bar{J} \cdot \text{curl } \bar{H}_m \, dV}{(k^2 - k_m^2) \iiint_V (\text{curl } \bar{H}_m)^2 \, dV} \text{ curl } \bar{H}_m. \qquad (12)$$

The magnetic field can similarly be expressed as

$$\bar{H}_0 = -\sum \frac{k_m^2}{k^2 - k_m^2} \frac{\iiint_V \bar{J}\cdot \text{curl } \bar{H}_m \, dV}{\iiint_V |\text{ curl } \bar{H}_m|^2 \, dV} \bar{H}_m$$

$$= -\sum \frac{\iiint_V \bar{J}\cdot \text{curl } \bar{H}_m \, dV}{(k^2 - k_m^2) \iiint_{V+V'} |\bar{H}_m|^2 \, dV} \bar{H}_m. \quad (13)$$

Here use has been made of the relationship between normalization integrals (see [1, eq. 28]).

$$\iiint_V |\text{ curl } \bar{H}_m|^2 \, dV = k_m^2 \iiint_{V+V'} |\bar{H}_m|^2 \, dV. \quad (14)$$

The expansions for $\bar{E}$ and $\bar{H}$ show that the fields reach infinite values at the resonant wavenumbers $k = k_m$. In fact, if we denote by $\Delta k$ the wavenumber difference $k - k_m$, $\bar{H}_0$ near resonance is given by

$$\bar{H}_0 \approx -\frac{\iiint_V \bar{J}\cdot \text{curl } \bar{H}_m \, dV}{2(\Delta k/k_m) \iiint_V |\text{ curl } \bar{H}_m|^2 \, dV} \bar{H}_m. \quad (15)$$

Expression (15) represents the magnetic field in the limit $N \to \infty$. For high, but *finite* values of $N$, two things will happen. First, the resonant frequency suffers a slight shift. This is a minor phenomenon, which will not be investigated here. Second, the infinite resonant amplitude is leveled off by the radiation losses, which introduce a finite $Q$, and a factor $[(\Delta k/k_m) - (j/2Q)]$ instead of $\Delta k/k_m$ in (15). Thus

$$\bar{H} \approx -\frac{\iiint_V \bar{J}\cdot \text{curl } \bar{H}_m \, dV}{2(\Delta k/k_m - j/2Q) \iiint_V |\text{ curl } \bar{H}_m|^2 \, dV} \bar{H}_m. \quad (16)$$

The numerator $\iiint_V \bar{J}\cdot \text{curl } \bar{H}_m \, dV$ expresses the coupling between the mode and the current distribution. For a source in the form of a concentrated electric dipole $\bar{P}_e$, for example,

$$\iiint \bar{J}\cdot \text{curl } \bar{H}_m \, dV = \text{curl } \bar{H}_m \cdot \iiint \bar{J} \, dV = j\omega \bar{P}_e \cdot \text{curl } \bar{H}_m$$

$$= j\frac{kc}{N} \bar{P}_e \cdot \text{curl } \bar{H}_m \quad (17)$$

where curl $\bar{H}_m$ is the value at the location of the dipole. For a concentrated magnetic dipole $\bar{P}_m$

$$\iiint \bar{J}\cdot \text{curl } \bar{H}_m \, dV = k_m^2 \bar{P}_m \cdot \bar{H}_m \quad (18)$$

where $\bar{H}_m$ is the value at the location of the dipole.

## III. EXCITATION BY INTERIOR SOURCES—VERIFICATION OF THE FORMULAS

### *A. Sphere Excited by a Magnetic Dipole*

Equations such as (12), (13), or (16) are new, and must therefore be checked against configurations for which an exact solution exists. We have performed this verification in two cases. First, we have investigated a dielectric sphere excited by a *magnetic* dipole located at its center [Fig. 2(a)] This configuration happens to be of technical interest, as the radiation resistance of the dipole at resonance peaks to a substantially higher value than in free space. This property allows one to match the very low impedance of electrically short antennas to their feeding generators. Considerable reduction of antenna dimensions compared with the usual half- and quarter-wave antennas can then be obtained [3], [4]. Our purpose, however, is not to dwell upon these aspects, but to solve the problem theoretically. Separation of variables yields a solution which is valid for arbitrary $N$, and which allows detailed investigation of the limit process $N \to \infty$. The electric field is purely azimuthal. Inside the resonator it is

$$E_\phi = -j\frac{kR_0P_m}{4\pi N} \sin\theta \left(\frac{\exp(-jkR)}{R^2} + jk\frac{\exp(-jkR)}{R}\right)$$

$$- j\frac{kR_0P_m}{4\pi N} B \sin\theta \left(\frac{\sin kR}{R^2} - \frac{k\cos kR}{R}\right). \quad (19)$$

Outside

$$E_\phi = -\frac{jkR_0}{4\pi N} P_m \sin\theta C \left(\frac{\exp[-j(kR/N)]}{R^2} + j\frac{k}{N}\frac{\exp[-j(kR/N)]}{R}\right). \quad (20)$$

We shall not evaluate coefficients $B$ and $C$ explicitly: they can easily be obtained from an application of the boundary conditions at $R = a$. The calculations show that $B$ and $C$ have a common denominator

$$D = \sin ka + j\frac{ka}{N}\sin ka + \frac{1}{N^2}(ka\cos ka - \sin ka). \quad (21)$$

Assume first that $N$ is infinite. For such case, the denominator is zero for $\sin ka = 0$, i.e., precisely at the resonant frequencies of the $\phi$-independent nonconfined modes given by [1, eqs. 63, 64]. The fields are infinite at resonance, and it can be verified that the values of $B$ and $C$ are exactly those predicted by (12) and (18). Let us now examine the effect of a high, but finite $N$. The most interesting phenomenon is the leveling off of the resonance peak. The exact values of the fields can again be obtained

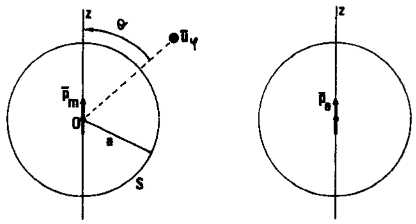


Fig. 2. Dielectric sphere excited by the following. (a) A magnetic dipole. (b) An electric dipole.

from a study of the coefficients $B$ and $C$. The common denominator in these expressions takes the form

$$a \cos k_m a \left[\Delta k + \frac{k_m}{N^2} + j\Delta k \frac{k_m a}{N}\right]. \tag{22}$$

This formula shows that the resonant frequency is shifted to a slightly different value, given by

$$k_m' = k_m\left(1 - \frac{1}{N^2}\right). \tag{23}$$

The frequency-excursion with respect to $k_m'$ is $\Delta k' = k - k_m'$. In terms of this increment, (22) can be written as

$$k_m a \cos k_m a \left[\frac{\Delta k'}{k_m} - j\frac{k_m a}{N^3}\right].$$

But the characteristic denominator for a damped resonance is of the form $[(\Delta k/k) - (j/2Q)]$. We conclude that the quality factor of the mode is given by

$$Q = \frac{N^3}{2k_m a} \tag{24}$$

which is precisely the value calculated by other means in [1, eq. 66]. There is therefore agreement with the factor $[(\Delta k/k_m) - (j/2Q)]$ appearing in the denominator of (16). Detailed calculations confirm the correctness of the other factors in (16). They also show that the fields peak to values proportional to $N^3$ (for $\bar{H}$) and $N^2$ (for $\bar{E}$) in the vicinity of the sphere, and to $N$ in the far field.

### B. Sphere Excited by an Electric Dipole

Let us now investigate the fields produced by an *electric* dipole in the center of the sphere [Fig. 2(b)]. Here (17) predicts that the confined modes only are excited, hence that the boundary surface acts as a magnetic wall. These predictions are confirmed by an exact analysis, which can be carried much as in the preceding paragraph. The *magnetic field* is azimuthal in this case, and is given by

$$H_\phi = j\frac{kc}{4\pi N} P_e \sin\theta\left(\frac{\exp(-jkR)}{R^2} + jk\frac{\exp(-jkR)}{R}\right) + j\frac{kc}{4\pi N} P_e \sin\theta B\left(\frac{\sin kR}{R^2} - k\frac{\cos kR}{R}\right) \tag{25}$$

in the dielectric, and

$$H_\phi = j\frac{kc}{4\pi N} P_e \sin\theta C\left(\frac{\exp[-j(kR/N)]}{R^2} + \frac{jk}{N}\frac{\exp[-j(kR/N)]}{R}\right) \tag{26}$$

outside the dielectric. The coefficients $B$ and $C$, which can be determined through the boundary conditions at $R = a$, have the common denominator

$$D = (ka\cos ka - \sin ka)\left(1 + j\frac{ka}{N} - \frac{1}{N^2} - \frac{jka}{N^3}\right) - \frac{k^3a^3\cos ka}{N^2} - j\frac{k^3a^3\sin ka}{N^3}. \tag{27}$$

For $N \to \infty$, resonances occur for $ka\cos ka - \sin ka = 0$, i.e., precisely for the resonant frequencies of the ($\phi$-independent) confined modes given in [1, eq. 67]. For $N$ high, but finite, the resonant frequency is shifted to a new value, obtained by setting the real part of the denominator equal to zero. This gives the new wavenumber

$$k_m' = k_m\left(1 - \frac{1}{N^2}\right). \tag{28}$$

A few calculations show that $D$ can be put in the form

$$D = -k_m^2a^2\sin k_m a\left(\frac{\Delta k'}{k_m} - \frac{j}{2Q}\right)$$

where

$$Q = \frac{N^5}{2k_m^3a^3}. \tag{29}$$

This is precisely the quality factor obtained in [1, eq. 70] by use of different methods. Detailed calculations confirm the correctness of the other factors in (16). Let us mention, in this respect, that the interior magnetic field peaks to values proportional to $N^4$. The radiated fields are proportional to $N$ (and are therefore of order $1/N^3$ with respect to the interior fields), and the radiated power is proportional to $N^2$.

## IV. SCATTERING BY DIELECTRIC RESONATORS—THE NONCONFINED MODES

In the present paragraph we consider a dielectric resonator immersed in an incident wave $(\bar{E}_i, \bar{H}_i)$. Our purpose is to evaluate the interior and exterior fields in the vicinity of a resonant frequency. The problem is of technical interest, as the scattering response at resonance can be utilized to measure the characteristics of the dielectric material [5]; and to obtain information on the size and shape of the particles (e.g., in a study of the properties of interstellar dust).

### A. Expansion of the Incident Field

Away from its sources, and in particular at the location of the dielectric body, the incident field satisfies

$$\operatorname{curl} \bar{H}_i = \frac{jk}{NR_0} \bar{E}_i$$

$$\operatorname{curl} \bar{E}_i = -j \frac{kR_0}{N} \bar{H}_i. \tag{30}$$

In and around the dielectric body (Fig. 3), we introduce the expansions

$$\bar{E}_i = \bar{E}_{i0} + \frac{\bar{E}_{i1}}{N} + \frac{\bar{E}_{i2}}{N^2} + \cdots$$

$$\bar{H}_i = \bar{H}_{i0} + \frac{\bar{H}_{i1}}{N} + \frac{\bar{H}_{i2}}{N^2} + \cdots. \tag{31}$$

For the plane wave $\bar{E}_i = A\bar{u}_x \exp[-j(kz/N)]$, for example, (Fig. 3)

$$\bar{E}_{i0} = A\bar{u}_x \qquad \bar{H}_{i0} = \frac{A}{R_0}\bar{u}_y$$

$$\bar{E}_{i1} = -jkzA\bar{u}_x \qquad \bar{H}_{i1} = -\frac{A}{R_0} jkz\bar{u}_y. \tag{32}$$

The individual terms in (31) satisfy the equations

$$\operatorname{curl} \bar{H}_{i0} = 0$$

$$\operatorname{curl} \bar{H}_{i1} = j\frac{k}{R_0}\bar{E}_{i0}$$

$$\operatorname{curl} \bar{E}_{i0} = 0$$

$$\operatorname{curl} \bar{E}_{i1} = -jkR_0\bar{H}_{i0}. \tag{33}$$

### *B. Expansion of the Scattered Field*

The total field $(E,H)$ does not behave like the eigenvectors $(\bar{H}_m, \operatorname{curl} \bar{H}_m)$ in the vicinity of the dielectric. It does not, in particular, decrease like $1/R^3$ away from the body. The set $(\bar{H}_m, \operatorname{curl} \bar{H}_m)$ is therefore not suitable to expand the total field outside the resonator. We recognize, however, (and this is a crucial remark) that the *additional* field produced by the presence of the dielectric, viz.,

$$\bar{e} = \bar{E} - \bar{E}_i$$

$$\bar{h} = \bar{H} - \bar{H}_i \tag{34}$$

has the same behavior as $(\bar{H}_m, \operatorname{curl} \bar{H}_m)$, and can therefore usefully be expanded in these eigenvectors. We write

$$\bar{h} = \sum \beta_m \bar{H}_m \tag{35}$$

where

$$\beta_m = \frac{\iiint_{V+V'} \bar{h}\cdot\bar{H}_m \, dV}{\iiint_{V+V'} |\bar{H}_m|^2 \, dV}. \tag{36}$$

The evaluation of $\beta_m$ rests on a knowledge of the equations

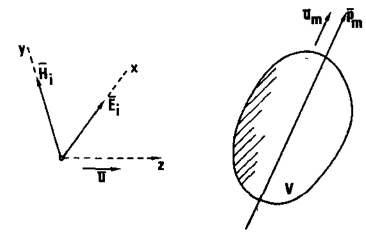


Fig. 3. Dielectric resonator in an incident wave. Excitation of nonconfined modes.

satisfied by $\bar{h}$. Outside the dielectric, they are the same as for the incident field, i.e., (33). Inside the dielectric, the total field satisfies

$$\operatorname{curl} \bar{E} = -j\frac{kR_0}{N}\bar{H}$$

$$\operatorname{curl} \bar{H} = \frac{jk}{R_0} N\bar{E} \tag{37}$$

and the incident field

$$\operatorname{curl} \bar{E}_i = -j\frac{kR_0}{N}\bar{H}_i$$

$$\operatorname{curl} \bar{H}_i = \frac{jk}{R_0 N}\bar{E}_i. \tag{38}$$

Subtraction gives

$$\operatorname{curl} \bar{e} = -j\frac{kR_0}{N}\bar{h}$$

$$\operatorname{curl} \bar{h} = \frac{jk}{R_0}N\bar{e} + \frac{jk}{R_0}N\left(1 - \frac{1}{N^2}\right)\bar{E}_i. \tag{39}$$

Equations for $\bar{e}$ and $\bar{h}$ alone can be obtained by elimination. They are

$$-\operatorname{curl}\operatorname{curl} \bar{e} + k^2\bar{e} = -k^2\bar{E}_i\left(1 - \frac{1}{N^2}\right)$$

$$-\operatorname{curl}\operatorname{curl} \bar{h} + k^2\bar{h} = -k^2\bar{H}_i\left(1 - \frac{1}{N^2}\right). \tag{40}$$

These equations are fundamental. The right-hand members are known, and can be regarded as sources for, respectively, the $\bar{e}$ and $\bar{h}$ fields. To evaluate the numerator in (36), we dot-multiply the equation for $\bar{h}$ with $\bar{H}_m$, and integrate. Thus

$$-\iiint_V \bar{H}_m\cdot\operatorname{curl}\operatorname{curl}\bar{h} + k^2\iiint_V \bar{h}\cdot\bar{H}_m\,dV$$

$$= -k^2\left(1 - \frac{1}{N^2}\right)\iiint_V \bar{H}_i\cdot\bar{H}_m\,dV. \tag{41}$$

Utilizing the methods applied in [1, eqs. 18–24] allows us to transform the first integral into

$$-\iiint_V \bar{H}_m \cdot \text{curl curl } \bar{h}\, dV$$

$$= -\iiint_V \bar{h} \cdot \text{curl curl } \bar{H}_m\, dV$$

$$+ \iint_S [(\bar{u}_n \times \bar{H}_m) \cdot \text{curl } \bar{h} - (\bar{u}_n \times \bar{h}) \cdot \text{curl } \bar{H}_m]\, dS$$

$$= -k_m^2 \iiint_V \bar{h} \cdot \bar{H}_m\, dV - \iint_S \psi_m \bar{u}_n \cdot \text{curl curl } \bar{h}\, dV$$

$$+ k_m^2 \iint_S \psi \frac{\partial \psi_m}{\partial n}\, dS. \tag{42}$$

To arrive at the last equation we have written $\bar{H}_m = \text{grad}\, \psi_m$ and $\bar{h} = \text{grad}\, \psi$ outside and on $S$. The last relationship implies that $\bar{h}$ has been replaced by its zeroth-order term, hence that terms of order $1/N$ and higher have been neglected. Further progress is made by replacing curl curl $\bar{h}$ by its value taken from (40). The surface integrals in (42) then add up to

$$-k^2 \iint_S \psi_m (\bar{u}_n \cdot \bar{H}_i)\, dS - k^2 \iint_S \psi_m \frac{\partial \psi}{\partial n}\, dS$$

$$+ k_m^2 \iint_S \psi \frac{\partial \psi_m}{\partial n}\, dS.$$

As in [1, eq. 23], the last two terms can be shown to stand for

$$(k_m^2 - k^2) \iiint_{V'} \bar{h} \cdot \bar{H}_m\, dV.$$

Collecting these various results leads to the following contribution of a nonconfined mode $\bar{H}_m$ to the expansion of $\bar{h}$

$$\frac{k^2}{k^2 - k_m^2} \frac{\iint_S \psi_m (\bar{u}_n \cdot \bar{H}_i)\, dS - \iiint_V \bar{H}_i \cdot \bar{H}_m\, dV}{\iiint_{V+V'} |\bar{H}_m|^2\, dV} \bar{H}_m. \tag{43}$$

The integrals in the numerator express the coupling between $\bar{H}_i$ and $\bar{H}_m$. An equivalent formulation in terms of $\bar{E}_i$ and curl $\bar{H}_m$ is also possible. We give it without proof

$$\iint_S \psi_m (\bar{u}_n \cdot \bar{H}_i)\, dS - \iiint_V \bar{H}_i \cdot \bar{H}_m\, dV$$

$$= -\frac{j}{kR_0} \iiint_V \bar{E}_{i1} \cdot \text{curl } \bar{H}_m\, dV. \tag{44}$$

Expression (43) clearly shows that resonances occur for $k = k_m$. For high but finite $N$, the fields near resonance are of the form

$$\bar{H}(\bar{r}) \approx \frac{\iint_S \psi_m (\bar{u}_n \cdot \bar{H}_i)\, dS - \iiint_V \bar{H}_i \cdot \bar{H}_m\, dV}{2(\Delta k/k_m - j/2Q) \iiint_{V+V'} |\bar{H}_m|^2\, dV} \bar{H}_m(\bar{r}). \tag{45}$$

## C. *Dielectric Resonator Immersed in a Plane Wave*

It is shown in [1, eq. 45] that the magnetic dipole moment of a nonconfined mode $\bar{H}_m$ is given by

$$\bar{p}_m = \iiint_V \bar{H}_m\, dV - \iint_S \psi_m \bar{u}_n\, dS = p_m \bar{u}_m. \tag{46}$$

Here $\bar{u}_m$ is the unit vector in the direction of the dipole moment. This expression allows one to write the coupling coefficient (44) very concisely when the incident wave is a plane wave. For such case, $\bar{H}_i$ is a constant field $\bar{H}_{i0}$ in the limit of very high $N$, i.e., for very low frequencies. It follows that (44) can be rewritten as $-\bar{p}_m \cdot \bar{H}_{i0}$. This value can now be inserted in (45). The peak field, reached for the resonance condition $\Delta k = 0$, is

$$\bar{H}_{\text{peak}} = -jQ \frac{\bar{p}_m \cdot \bar{H}_{i0}}{\iiint_{V+V'} |\bar{H}_m|^2\, dV} \bar{H}_m = -j \frac{6\pi N^3}{k_m^3} \frac{\bar{p}_m \cdot \bar{H}_{i0}}{|\bar{p}_m|^2} \bar{H}_m \tag{47}$$

where use has been made of the value of $Q$ derived in [1, eq. 43]. The formula for $\bar{H}_{\text{peak}}$ is seen to be remarkably simple. An even more remarkable value is obtained for the magnetic moment of the resonator at resonance, which is, from (46) and (47),

$$(\bar{P}_m)_{\text{peak}} = -j \frac{6\pi N^3}{k_m^3} (\bar{H}_{i0})_m = -j \frac{3}{4\pi^2} \lambda_0^3 (\bar{H}_{i0})_m. \tag{48}$$

In this equation, $(\bar{H}_{i0})_m$ is the component of the incident magnetic field along the direction of $\bar{p}_m$, and $\lambda_0$ is the wavelength *in vacuo*. Formula (48) deserves additional comments. It shows that, in the limit of very high $\epsilon_r$, the peak magnetic moment at resonance is independent of $\epsilon_r$, the mode number, and the shape of the scatterer. These parameters, however, influence the value of $Q$, i.e., the *shape* of the resonance curve. They also determine the size of the resonator, which results from the equation

$$L = \frac{\lambda_0}{2\pi N} (kL)_{\text{mode}} \tag{49}$$

where $(kL)_{\text{mode}}$ is a characteristic value for each mode. Notice also, from (48), that only the *direction* of $\bar{p}_m$ must be known to determine the magnetic moment. Information as to the *magnitude* of $\bar{p}_m$ is not necessary for the purpose.

Knowledge of the dipole moment allows calculation of the scattered power and of the scattering cross section at resonance. Here again, the formula is of the utmost simplicity

$$\sigma_{sc} = \frac{6\pi N^2}{k_m^2} \frac{|\bar{u}_m \cdot \bar{H}_{i0}|^2}{|\bar{H}_{i0}|^2} = \frac{3}{2\pi} \lambda_0^2 \frac{|\bar{u}_m \cdot \bar{H}_{i0}|^2}{|\bar{H}_{i0}|^2}. \quad (50)$$

The scattering cross section is maximum when the incident magnetic field is parallel with the natural dipole moment of the mode. At 10 GHz, for example, this maximum value is 4.3 cm², irrespective of the shape, mode number, and dielectric constant of the scatterer. These remarkable properties hold only in the domain of validity of our hypotheses, i.e., for free-space wavelengths $\lambda_0$ which are *large* compared with the characteristic dimension $L$. Equation (50) obviously implies that the mode under consideration has a nonzero magnetic moment.

## V. SCATTERING BY DIELECTRIC RESONATORS—THE CONFINED MODES

### *A. Expansion of the Scattered Field*

Evaluation of the expansion coefficient $\beta_m$, as given in (36), is more delicate for the confined modes. The right-hand side in (41) is now a term of order $1/N$. To prove this assertion, let us show that the contribution from $\bar{H}_{i0}$ vanishes. From (33), $\bar{H}_{i0}$ can be expressed as grad $\psi_{i0}$. Therefore

$$\iiint_V \bar{H}_{i0} \cdot \bar{H}_m \, dV = \iiint_V \operatorname{grad} \psi_{i0} \cdot \bar{H}_m \, dV$$

$$= \iint_S (\bar{u}_n \cdot \psi_{i0} \bar{H}_m) \, dS - \iiint_V \psi_{i0} \operatorname{div} \bar{H}_m \, dV.$$

This expression is zero because $\bar{H}_m$ vanishes along $S$, and has zero divergence in $V$. As a result, the right-hand member of (41) becomes

$$-k^2 \iiint_V \bar{H}_i \cdot \bar{H}_m \, dV = -\frac{k^2}{N} \iiint_V \bar{H}_{i1} \cdot \bar{H}_m \, dV. \quad (51)$$

This result has an immediate implication: the surface integral in (42) must be evaluated very carefully, up to terms in $1/N$. As $\bar{H}_m$ vanishes along $S$, this integral can be written as

$$I = -\iint_S (\bar{u}_n \times \bar{h}) \cdot \operatorname{curl} \bar{H}_m \, dS.$$

Replacing curl $\bar{H}_m$ by its value in terms of $\bar{E}_{m1}$, as given in (9), yields

$$I = -\frac{jk}{R_0} \iint_S (\bar{E}_{m1} \times \bar{u}_n) \cdot \bar{h} \, dS$$

$$= -\frac{jk}{R_0} \iint_S (\bar{E}_{m1}' \times \bar{u}_n) \cdot \bar{h} \, dS.$$

But $\bar{E}_{m1}'$, the field outside $S$, is irrotational, and can therefore be written as grad $\phi_{m1}'$. Utilizing classical vector relationships [3] allows one to further transform $I$ as

$$I = -\frac{jk}{R_0} \iint_S \operatorname{grad}_s \phi_{m1}' \cdot (\bar{u}_n \times \bar{h}) \, dS$$

$$I = \frac{jk}{R_0} \iint_S \phi_{m1}' \operatorname{div}_s (\bar{u}_n \times \bar{h}) \, dS$$

$$= \frac{jk}{R_0} \iint_S \phi_{m1}' \operatorname{div}_s (\bar{u}_n \times \bar{h}') \, dS$$

$$= -\frac{jk}{R_0} \iint_S \phi_{m1}' (\bar{u}_n \cdot \operatorname{curl} \bar{h}') \, dS.$$

The exterior field $\bar{h}'$ satisfies the equations given in (33) for the incident field. In consequence, curl $\bar{h}'$ can be replaced by $(1/N)$ curl $\bar{h}_1'$, and

$$I = -\frac{jk}{R_0 N} \iint_S \phi_{m1}' \bar{u}_n \cdot \operatorname{curl} \bar{h}_1' \, dS = \frac{k^2}{N R_0^2} \iint_S \phi_{m1}' e_{0n}' \, dS.$$

The outside electric field $\bar{e}_0'$ can be expressed as grad $\phi_0'$, where $\phi_0'$ is regular at infinity. Combining this fact with a basic reciprocity property proved in [1, eq. 23] gives the further transformation

$$I = \frac{k^2}{N R_0^2} \iint_S \phi_{m1}' \frac{\partial \phi'_0}{\partial n} \, dS = \frac{k^2}{N R_0^2} \iint_S \phi_0' \frac{\partial \phi_{m1}'}{\partial n} \, dS. \quad (52)$$

The factor $(\partial \phi_{m1}/\partial n)$ has been encountered in the general theory of the confined modes (see [1, eq. 56]), where it has been denoted by $E_{1n}'$). The factor $\phi_0'$ can be replaced by $-\phi_{0i}$. It is indeed apparent, from (43), that the total zeroth-order electric field must vanish in $V$, hence that $\bar{e}_0$ must be equal to $-\bar{E}_{i0}$ inside the resonator. Therefore

$$(\bar{e}_0')_{\text{tang}} = \operatorname{grad}_s \phi_0' = (\bar{e}_0)_{\text{tang}} = -\operatorname{grad}_s \phi_{i0}$$

where $\phi_{i0}$ is the potential from which $\bar{E}_{i0}$ can be derived. These considerations lead to

$$I = -\frac{k^2}{N R_0^2} \iint_S \phi_{0i} E_{n1}' \, dS. \quad (53)$$

Our main hurdle has now been taken. With this value of the surface integral, the determination of the expansion coefficient $\beta_m$ becomes easy. We obtain, for the contribution of a confined mode to the expansion of $\bar{h}$,

$$-\frac{k^2}{k^2 - k_m^2} \frac{\iiint_V \bar{H}_{i1} \cdot \bar{H}_m \, dV - (1/R_0^2) \iint_S \phi_{0i} E_{n1}' \, dS}{N \iiint_V |\bar{H}_m|^2 \, dV}. \quad (54)$$

In the vicinity of the mode resonance, in particular, the magnetic field is of the form

$$\bar{H} = \frac{\iiint_V \bar{H}_{i1} \cdot \bar{H}_m \, dV - (1/R_0^2) \iint_S \phi_{0i} E_{n1}' \, dS}{2N(\Delta k/k_m - j/2Q) \iiint_V |\bar{H}_m|^2 \, dV} \bar{H}_m. \quad (55)$$

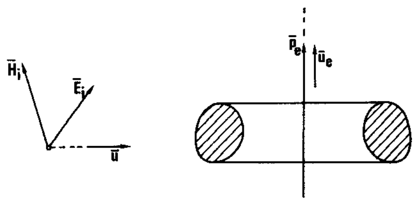

Fig. 4. Body of revolution in an incident wave. Excitation of confined modes.

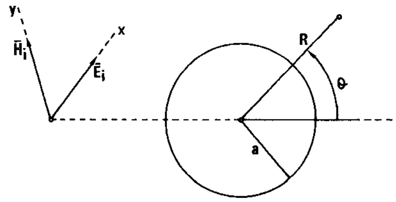

Fig. 5. Dielectric sphere in incident plane wave.

### B. Dielectric Resonator Immersed in a Plane Wave

The configuration of interest is sketched in Fig. 4. A confined mode $\bar{H}_m = \beta \bar{u}_\phi$ possesses an electric dipole moment (see [1, eq. 57])

$$\bar{p}_e = \left[\frac{\epsilon_0}{N}\iint_S E_{1n}'z\,dS - j\frac{\pi k}{Nc}\iint_{S_m}\beta r^2\,dS\right]\bar{u}_e \quad (56)$$

where $\bar{u}_e$ is a unit vector along the axis of revolution. Careful evaluation of (54), where $\bar{H}_{i1}$ and $\phi_{0i}$ are replaced by the values pertinent to a plane wave [see (32)], leads to the remarkable result

$$\iiint_V \bar{H}_{i1}\cdot\bar{H}_m\,dV - \frac{1}{R_0^2}\iint_S \phi_{0i}E_{n1}'\,dS = -\frac{N}{\mu_0}\bar{E}_{i0}\cdot\bar{p}_e. \quad (57)$$

The field near resonance is therefore

$$\bar{H} = \frac{\bar{E}_{i0}\cdot\bar{p}_e}{2\mu_0(\Delta k/k_m - j/2Q)\iiint_V |\bar{H}_m|^2\,dV}\bar{H}_m. \quad (58)$$

From the value of $Q$ given in (58) of [1], the peak value of $\bar{H}$ is

$$(\bar{H})_{\text{peak}} = j\frac{6\pi N^3\epsilon_0}{k_m^3}\frac{\bar{E}_{i0}\cdot\bar{p}_e}{|\bar{p}_e|^2}\bar{H}_m. \quad (59)$$

The remarkable simplicity of the formulas for the nonconfined modes is seen to carry over to the confined modes. The induced electric dipole moment at resonance is

$$(\bar{P}_e)_{\text{peak}} = -j\frac{6\pi N^3\epsilon_0}{k_m^3}(\bar{E}_{i0})_e \quad (60)$$

where $(\bar{E}_i)_e$ is the component of $\bar{E}_i$ along the axis of revolution. The peak scattering cross section turns out to be

$$\sigma_{sc} = \frac{6\pi N^2}{k_m^2}\frac{|\bar{E}_{i0}\cdot\bar{u}_e|^2}{|\bar{E}_{i0}|^2} = \frac{3}{2\pi}\lambda_0^2\frac{|\bar{E}_{ie}|^2}{|\bar{E}_i|^2}. \quad (61)$$

## VI. SCATTERING BY A DIELECTRIC SPHERE—CONFIRMATION OF THE GENERAL THEORY

### A. Generalities

To check the validity of our previous analysis, and in particular the correctness of the main results (45)–(50), (55)–(61), we have considered the example of a dielectric sphere immersed in a plane incident wave (Fig. 5). Here separation of variables gives a solution for arbitrary $N$ [2], [6]. Careful calculations (to be outlined below) confirm that the general formulas yield correct results here. Our concern lies, as usual, with the limit case $N \to \infty$. Results for a few finite values of $N$ (real and complex) can be found in a recent article by Affolter and Eliasson [7].

The method of solution proceeds by writing the fields as

$$\bar{E} = \text{curl curl}\,(vR\bar{u}_R) - j\omega\mu\,\text{curl}\,(wR\bar{u}_R)$$
$$\bar{H} = j\omega\epsilon\,\text{curl}\,(vR\bar{u}_R) + \text{curl curl}\,(wR\bar{u}_R). \quad (62)$$

Different couples of functions $(v,w)$ are used for the incident plane wave, the scattered field, and the field in the dielectric. The $\boldsymbol{v}$ and $\boldsymbol{w}$ functions are infinite series involving Legendre polynomials in $\cos\theta$ and spherical Bessel and Hankel functions in $kR$. For very high values of $N$, the dominant terms in (62) are, for the scattered field (Fig. 5)

$$v_{sc} = j\frac{ka^3}{N}\sin\theta\cos\phi\left[1 - j\frac{N}{kR}\right]\frac{\exp[-j(kR/N)]}{R}A$$

$$w_{sc} = j\frac{ka^3}{R_0N}\sin\theta\sin\phi\left[1 - j\frac{N}{kR}\right]\frac{\exp[-j(kR/N)]}{R}B. \quad (63)$$

The incident wave is the plane wave described by (32). Full expressions for $A$ and $B$ will not be given here. We shall only mention that $A$ has a denominator

$$D_1 = (\sin ka - ka\cos ka)\cdot\left(1 - \frac{1}{N^2} - \frac{k^2a^2}{2N^2} - \frac{k^2a^2}{2N^4} + j\frac{2k^3a^3}{3N^3} + j\frac{k^3a^3}{2N^5}\right) + k^2a^2\sin ka\left(\frac{1}{N^2} + \frac{k^2a^2}{2N^4} - j\frac{k^3a^3}{3N^5}\right) \quad (64)$$

and that the denominator of $B$ is

$$D_2 = \sin ka\left(1 - \frac{1}{N^2} + \frac{k^2a^2}{2N^2} + j\frac{ka}{N^3} - j\frac{k^3a^3}{3N^3}\right) + ka\cos ka\left(\frac{1}{N^2} - j\frac{ka}{N^3}\right). \quad (65)$$

### B. Confined Modes

Function $v_{sc}$ generates the field of an *electric dipole*, the magnetic field of which is transverse to the radial direction. Resonances occur for

$$\sin ka - ka \cos ka = 0 \tag{66}$$

when $N$ is infinite. This is the resonant condition for the confined modes described in [1, eq. 67]. For high, but finite values of $N$, a more careful analysis of the denominator is required. Setting its real part equal to zero gives the actual resonant frequency, and consideration of its imaginary part yields $Q$. The values obtained by this procedure turn out to be in perfect agreement with (28) and (29). At each resonant frequency, $A$ peaks to a value

$$A_{\max} = -j \frac{3N^3}{2k_m^3a^3} \tag{67}$$

from which the far field is found to be

$$\bar{H} = A \frac{k_m^2a^3}{R_0N^2} (\bar{u}_R \times \bar{u}_x) \frac{\exp[-j(k_mR/N)]}{R}.$$

This is the field produced by an electric dipole

$$\bar{P}_e = +4\pi a^3 A \epsilon_0 \bar{u}_x. \tag{68}$$

The dipole moment at resonance is

$$(\bar{P}_e)_{\max} = -j6\pi \frac{N^3}{k_m^3} \epsilon_0 \bar{u}_x. \tag{69}$$

This value, and the resulting value of the scattering cross section, are in complete agreement with the predictions of (60) and (61)

### C. Nonconfined Modes

Function $w_{sc}$ generates the field of a *magnetic dipole*, and is responsible for the resonances of the nonconfined modes. A careful study of the denominator of $B$, as given in (65), confirms the values of resonant frequency and $Q$ obtained in (23) and (24). At resonance, $B$ reaches a peak value

$$B_{\max} = -j \frac{3}{2} \frac{N^3}{k_m^3a^3} \tag{70}$$

which corresponds to a far field

$$\bar{E} = -B \frac{k_m^3a^3}{N^2} (\bar{u}_R \times \bar{u}_y) \frac{\exp[-j(k_mR/N)]}{R}. \tag{71}$$

This is the field produced by a magnetic dipole

$$\bar{P}_m = \frac{1}{R_0} 4\pi a^3 B \bar{u}_y. \tag{72}$$

At a resonance frequency, $\bar{P}_m$ reaches a value

$$(\bar{P}_m)_{\max} = -j \frac{6\pi}{R_0} \frac{N^3}{k_m^3} \bar{u}_y \tag{73}$$

which is in complete agreement with the value obtained in (48) for a scatterer of arbitrary shape.

### D. The Sphere Between Resonances

Between resonances the sphere can be replaced, as a scatterer, by the induced dipole moments (68) and (72). The appropriate values for the coefficients can be derived from the general formulas. One finds, for very high $N$,

$$A = 1$$

$$B = \frac{3}{2}\left(\frac{1}{k^2a^2} - \frac{\cos ka}{ka \sin ka}\right) - \frac{1}{2}. \tag{74}$$

The scattering cross section of the sphere is

$$\sigma_{sc} = \frac{8\pi}{3} k_0^4a^6(A^2 + B^2) = \frac{8}{3} \frac{k^4a^4}{N^4} \pi a^2(A^2 + B^2). \tag{75}$$

This value is orders of magnitude lower than the value at resonance, as given by (50) and (61). A numerical example will illustrate the point. Consider a sphere of $\epsilon_r = 100$ resonating at *10 GHz* in the lowest nonconfined mode ($ka = \pi$). The radius of the sphere is then 1.5 mm, and the scattering cross section at resonance is 4.3 cm². At *9 GHz*, $A = 1$ and $B = 0.82$. The scattering cross section is now $3 \times 10^{-3}$ cm².

Formulas (74) allow us to examine the behavior of the sphere at very low frequencies. Well below the first resonance, when the diameter of the sphere is small with respect to the wavelength in the dielectric, the dipole coefficients are $A = 1$ and $B \approx 1/30\ (ka)^2$. The magnetic dipole moment is therefore negligible, and the electric dipole moment has the (static) value corresponding to a sphere of very high $\epsilon_r$ (which is also the value for a perfectly conducting sphere). Let the frequency increase monotonically. The magnetic moment $\bar{P}_m$ increases slowly until, for $ka = \pi$, it reaches a very high value at the first "nonconfined" resonance. For $ka = 4.49$, the first "confined" resonance is excited, and the *electric* dipole moment reaches a high value. Higher frequencies give rise to successive higher order resonances. In between resonances, however, $A$ remains equal to one. This result, which holds for sufficiently high values of $N$, implies that the electric dipole moment keeps the (static) value associated with a perfect conductor.

Curves for the scattering cross section of the sphere can be found in the literature. Affolter and Eliasson, for example [7], give computed data for a sphere of diameter 4 mm and dielectric constant $\epsilon_r = 100$ (Fig. 6). The predictions of our asymptotic formulas are in good agreement with their results. From (50) and (61), indeed, the scattering cross section for a dipole mode is seen to satisfy

$$\frac{\sigma_{sc}}{\pi a^2} = \frac{6\epsilon_r}{(k_m a)^2}. \tag{76}$$

For the magnetic dipole mode ($k_m a = \pi$), which resonates at 7.5 GHz in the limit $N \rightarrow \infty$, we expect

$$\frac{\sigma_{sc}}{\pi a^2} = \frac{600}{\pi^2} = 60.8. \tag{77}$$

For the lowest electric dipole mode ($k_m a = 4.49$), which resonates at 10.7 GHz in the limit $N \rightarrow \infty$, we expect

$$\frac{\sigma_{sc}}{\pi a^2} = \frac{600}{(4.49)^2} = 29.8. \tag{78}$$

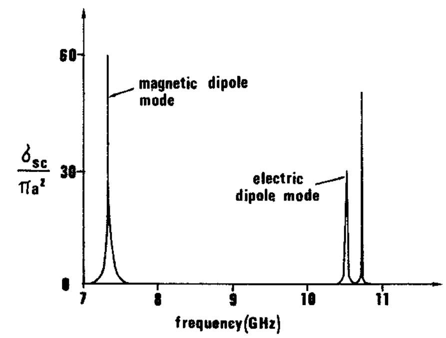


Fig. 6. Scattering cross section of a sphere as a function of frequency (from [7]).

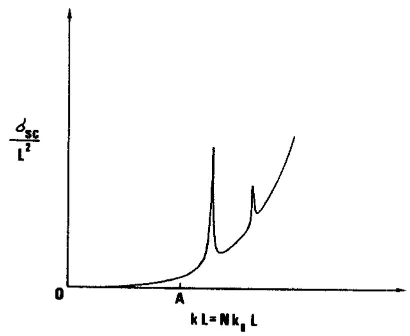


Fig. 7. Typical variation of the scattering cross section of a dielectric resonator as a function of frequency.

Both (asymptotic) results are in good agreement with Fig. 6. Notice that the radar cross section of the sphere is 1.5 $\sigma_{sc}$ (1.5 being the gain of the induced dipole). We therefore expect

$$\frac{\sigma_{\text{rad}}}{\pi a^2} = 0.912\epsilon_r \quad \text{for the magnetic dipole}$$

$$= 0.447\epsilon_r \quad \text{for the electric dipole.} \tag{79}$$

These results are in good agreement with the published data of Burr and Lo [5, fig. 2].

The resonance peaks in Fig. 6 are sharp because the value of $N$ is (relatively) high. For low values of $N$, the resonances are less pronounced. This is confirmed by published data, given for $N = 1.44$ [7], [8], where the first two resonances appear as mere bumps in the curve $\sigma_{sc}$ versus frequency.

The outlined behavior of the sphere is quite general, and is typical for resonators of arbitrary shape. The curve for $\sigma_{sc}$ has the general appearance depicted in Fig. 7, where $L$ is a typical dimension of the resonator. The curve starts with a Rayleigh region $OA$ in which $\sigma_{sc}/L$ is proportional with $(kL)^4$, and then shows resonance peaks superimposed on a smooth curve of the type encountered for perfectly conducting scatterers. Between resonances the fields in the resonator must remain bounded for $N \rightarrow \infty$. From (37), this condition implies that the electric field must be zero in $V$ (Fig. 1). Continuity of $\bar{E}_{\tan}$ therefore requires $\bar{E}$ to be perpendicular to $S$, which is precisely the behavior associated with a perfect conductor. The polarization charges $\rho_s$ are consequently identical for the metallic and high-dielectric scatterers, which implies that the *electric* dipole moments

$$\bar{P}_e = \iint_S \rho_s \bar{r}\, dS \tag{80}$$

are also equal. An analog property does not exist for the *magnetic* dipole moments. The essential reason is that $\bar{H}$ has a nonzero normal component on the dielectric, while $H_n$ must vanish on a perfect conductor.

It is shown in Section IV that the fields at resonance are proportional with $Q_{\text{rad}}$, and therefore that the peak scattering cross sections are proportional with $Q_{\text{rad}}^2$. When the material is lossy, the formulas are still valid provided $Q_{\text{rad}}$ is replaced by $Q_{\text{tot}} = (Q_{\text{rad}}^{-1} + Q_{\text{diel}}^{-1})^{-1}$. In particular, the radar cross section must be given by

$$\frac{\sigma_{\text{rad}}}{\pi a^2} = \left(\frac{Q_{\text{tot}}}{Q_{\text{rad}}}\right)^2 \frac{\sigma_0}{\pi a^2} \tag{81}$$

where $\sigma_0$ is the value for the lossless material. The validity of (81) can be checked on the published results of Burr and Lo [5]. These authors give data for dielectric spheres of permittivity extending only to $\epsilon_r = 50$, a value for which our asymptotic formulas should not be very accurate. The agreement turns out to be satisfactory, however, as seen from Table I relative to a dielectric constant

$$\epsilon_r = \epsilon_r' + j\epsilon_r'' = \epsilon_2'\left(1 - \frac{j}{Q}\right) = 50 - j0.2.$$

TABLE I

| | $(Q_{\text{rad}})_{\text{pert}}$ | $Q_{\text{diel}}$ | $Q_{\text{tot}}$ | $\frac{\sigma_0}{\pi a^2}$ | $(\frac{\sigma_{\text{rad}}}{\pi a^2})_{\text{pert}}$ | $(\frac{\sigma_{\text{rad}}}{\pi a^2})$ exact |
|---|---|---|---|---|---|---|
| magnetic dipole | 56.27 | 250 | 45.9 | 45.6 | 30.38 | 29.85 |
| electric dipole | 97.65 | 250 | 70.2 | 22.35 | 11.56 | 14.75 |

In Table I the asymptotic values, computed from (79) and (81), are denoted by the subscript "pert."

The results for the magnetic dipole are better than those for the electric dipole. This is to be expected, as the sphere is smaller with respect to $\lambda_0$ at the magnetic resonance than at the electric resonance.

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